United Kingdom
Mathematics Trust

Junior Mathematical Olympiad

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Solutions

Section A

A1. What is the value of $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6}$?

SOLUTION $\boxed{\frac{37}{60}}$

$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} = \frac{30 - 40 + 45 - 48 + 50}{60} = \frac{37}{60}$.

A2. Seven consecutive odd numbers add up to 105. What is the largest of these numbers?

SOLUTION $\boxed{21}$

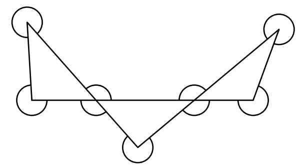
Let the fourth odd number be x. Then $(x-6)+(x-4)+(x-2)+(x)+(x+2)+(x+4)+(x+6) = 105$. This simplifies to $7x = 105$, which gives $x = 15$. The largest of the seven consecutive odd numbers is $15 + 6 = 21$.

A3. In a class, 55% of students scored at least 55% on a test. 65% of students scored at most 65% on the same test. What percentage of students scored between 55% and 65% (inclusive) on the test?

SOLUTION $\boxed{20\%}$

As 55% of students scored at least 55% on the test, 45% of the students scored less than 55%. Also, 65% of students scored 65% or less on the test, so the percentage of students who scored between 55% and 65% (inclusive) is 65% – 45% = 20%.

A4. What is the sum of the marked angles in this diagram?



SOLUTION $\boxed{1980°}$

First we consider the sum of the angles around each of the seven vertices of the three triangles, which is $7 \times 360° = 2520°$. The sum of the marked angles is the previous sum minus the sum of the interior angles of the three triangles, which is $2520° - 3 \times 180° = 2520° - 540° = 1980°$.

A5. Consider the six-digit multiples of three with at least one of each of the digits 0, 1 and 2, and no other digits. What is the difference between the largest and the smallest of these numbers?

SOLUTION **122 208**

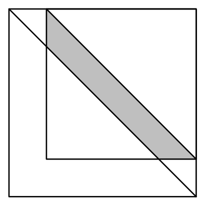
Multiples of three have a digit sum which is divisible by three. The six-digit numbers we are considering must have the digits 0, 1 and 2 occurring at least once. These three digits sum to 3, which means the three remaining digits must also sum to a multiple of three. To create the largest possible number, we want to maximise the digits used from left to right. To create the smallest possible number, we want to minimise the digits used from left to right but the first digit must not be 0 or else the number does not have six digits. For the largest such six-digit number we can use another three digit 2s and for the smallest we can use another three digits 0s. The largest possible number formed of the relevant digits is 222 210 and the smallest possible number formed of the relevant digits is 100 002. The difference we want is 222 210 − 100 002 = 122 208.

A6. Two positive numbers a and b, with $a > b$, are such that twice their sum is equal to three times their difference. What is the ratio $a : b$?

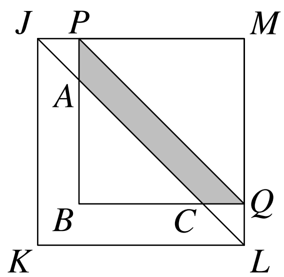
SOLUTION **5 : 1**

The information given can be written as $2(a + b) = 3(a - b)$, which upon expansion of the brackets gives $2a + 2b = 3a - 3b$. After rearrangement we have that $5b = a$ and so the ratio $a : b$ is $5 : 1$.

A7. The diagram on the right shows a 4 by 4 square placed on top of a 5 by 5 square, so that they have one vertex in common as shown. One diagonal of each square is also drawn. What is the area of the shaded region that is inside the 4 by 4 square and between the two diagonals?



SOLUTION **3.5**



Referring to the diagram on the right, JL is a diagonal of the square $JKLM$ therefore $\angle PJA = 45°$. The given configuration of the two squares means that $\angle APJ = 90°$ therefore $\angle JAP = 180° - 45° - 90° = 45°$. Hence triangle APJ is isosceles and $AP = JP$. As JP is the difference between the side lengths of the two squares, then $AP = JP = 5 - 4 = 1$. Similarly, we can deduce that $CQ = 1$. The shaded area is equal to the area of triangle PBQ minus the area of triangle ABC. As $AB = BC = 4 - 1$, this is $\frac{1}{2} \times 4 \times 4 - \frac{1}{2} \times 3 \times 3 = 3.5$ square units.

A8. The sum of the numbers 1 to 123 is 7626. One number is omitted so that the sum is now a multiple of 4. How many different numbers could be omitted?

SOLUTION **31**

7626 is two more than a multiple of four, which means for the reduced sum to be a multiple of four then the number omitted must also be two more than a multiple of four. The sequence of relevant numbers that could be removed is 2, 6, 10, . . . , 122. If we add two to each of these and then divide by four we get the sequence 1, 2, 3, . . . , 31. This shows that we could omit 31 different numbers.

A9. Dividing 52 by 12 gives 4 remainder 4. What is the sum of all the numbers for which dividing by 12 gives a whole number answer which is the same as the remainder?

SOLUTION **858**

Using the example given we can write $52 = 12\times4+4 = 13\times4$, and so in general we are looking for numbers of the form $12m + m = 13m$ where m is any possible remainder when dividing by twelve. When dividing by twelve the possible remainders are 0, 1, 2, ..., 11. The sum we want is $13\times0+13\times1+13\times2+\ldots+13\times11$. This can be rewritten as $13(0+1+2+\ldots11) = 13\times66$ and so the desired sum is 858.

A10. Farmer Alice has an alpaca, a cat, a dog, a gnu and a pig. She also has five fields in a row. She wants to put one animal in each field, but doesn't want to put two animals in adjacent fields if their names contain the same letter. In how many different ways can she place her animals?

SOLUTION **4**

The pig shares a letter with three other animals (alpaca, dog and gnu) and so it must have only one neighbour. This means that the pig must go at either end of the row and must be adjacent to the cat. The dog and gnu cannot be neighbours as they share the letter g, which means that the alpaca must be placed between them. The dog or the gnu can be placed next to the cat. This means that there are two choices for where the pig is placed, and for each of these choices there are two choices for which animal is placed next to the cat, and so in total there are $2\times2 = 4$ different arrangements.

Section B

B1. The sum of two numbers is 90.
40% of the first number is 15 more than 30% of the second number.

Find the two numbers.

SOLUTION

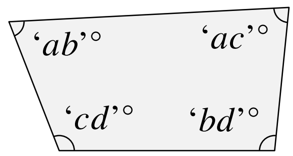
Let the first number be a. Then the second number is $90 - a$. Therefore $0.4a = 0.3(90 - a) + 15$.

Expanding the bracket gives $0.4a = 27 - 0.3a + 15$. Multiplying the equation by 10 gives $4a = 270 - 3a + 150$.

After rearrangement we have $7a = 420$. Therefore $a = 60$ and the two numbers are 60 and 30.

B2. In a certain quadrilateral, the four angles are each two-digit numbers. These four numbers can be placed in the 2 by 2 grid shown, with one digit in each cell.

Find all the possibilities for the set of four angles.



a	b
c	d

Solution

Throughout this solution we will assume that $b \geq c$ because the problem remains unchanged if we were to swap b and c.

It is useful to note that a two-digit number 'xy' can be written algebraically as $10x + y$ and that the sum of the four interior angles of a quadrilateral is 360°.

Combining the previous two facts we see that $10a + b + 10a + c + 10b + d + 10c + d = 360$. This can be simplified to give:

$$20a + 11b + 11c + 2d = 360. \tag{1}$$

If we assume that $a \leq 7$, then the maximum interior angle sum we can achieve is $20 \times 7 + 11 \times 9 + 11 \times 9 + 2 \times 9 = 356 < 360$ (where $a = 7$ and $b = c = d = 9$). As this is less than 360 we know that $a \geq 8$, and so a (which must be a digit) may only equal 8 or 9. We will consider each of these two cases separately.

If $a = 8$ then it must be that $b = 9$, or else the four interior angles are each less than 90° and the interior angle sum would be less than 360°. Substituting $a = 8$ and $b = 9$ into (1) and then simplifying gives $11c + 2d = 101$. The largest possible value of $2d$ is $2 \times 9 = 18$, and so $11c$ must be at least $101 - 18 = 83$. This means that c must be at least 8. If $c = 8$ then d would not be a whole number and so not a digit. If $c = 9$ then $d = 1$.

If $a = 9$ then b must be equal to 8 or 9, or else the maximum interior angle sum we can achieve is $20 \times 9 + 11 \times 7 + 11 \times 7 + 2 \times 9 = 352 < 360$ (where $a = 9$, $b = c = 7$ and $d = 9$).

If $a = 9$ and $b = 8$, substituting into (1) and then simplifying gives $11c + 2d = 92$. This means that $11c$ must be at least $92 - 18 = 74$ and so c is at least 7. If $c = 7$ then d is not a whole number and so not a digit. If $c = 8$ then $d = 2$. We know that c cannot equal 9 because at the start we assumed it was less than or equal to b.

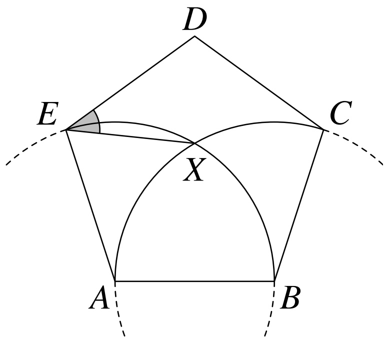
If $a = 9$ and $b = 9$, substituting into (1) and then simplifying gives $11c + 2d = 81$. This means that $11c$ must be at least $81 - 18 = 63$ and so c is at least 6. If $c = 6$ or 8 then d is not a whole number and so not a digit. If $c = 7$ then $d = 2$. We know that c cannot equal 9 because then d would be negative.

There are three sets of four angles that satisfy the problem: $\{89, 89, 91, 91\}$, $\{98, 98, 82, 82\}$ and $\{99, 97, 92, 72\}$.

B3. You start with a regular pentagon $ABCDE$. Then you draw two circles: one with centre A and radius AB, and the other with centre B and radius BA. Let the point inside the pentagon at which these two circles intersect be X.

What is the size of $\angle DEX$?

SOLUTION



As the pentagon $ABCDE$ is regular we know that $AE = AB = BC$. Hence B and E lie on the circle with centre A and radius AB, and A and C lie on the circle with centre B and radius BA.

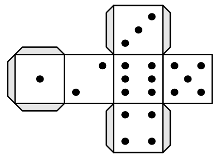
AX is a radius of the circle centre A, BX is a radius of the circle centre B, AB is a side length of the regular pentagon and so these three lengths are equal. Hence triangle AXB is equilateral and $\angle XAB = 60°$.

Given that $\angle EAB$ is an interior angle of a regular pentagon, it follows that $\angle EAX = \angle EAB - \angle XAB = (\frac{540}{5})° - 60° = 108° - 60° = 48°$.

$AE = AX$ as they are both radii of the circle centre A. Hence triangle AEX is isosceles. It follows that $\angle XEA = \angle AXE = (\frac{180-48}{2})° = 66°$.

As $\angle DEA$ is an interior angle of a regular pentagon, it follows that $\angle DEX = \angle DEA - \angle XEA = 108° - 66° = 42°$.

B4. Seth creates n standard dice by folding up n identical copies of the net shown. He then repeatedly puts one on top of another until there are none left, creating a vertical tower.



For each of the four vertical walls of the tower, he finds the total number of dots that are visible.

Given that the four totals calculated are all odd, what are the possible values for n?

Solution

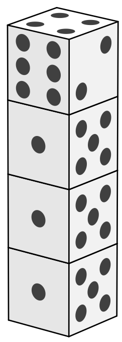
Consider a tower made up of n dice. Let the total number of dots on the vertical wall facing us be S. Let the total number of dots on the wall opposite this, which is facing away from us, be T.

Looking at the diagram we can see that each pair of opposite faces on a completed net will have a total of seven dots (this is true of all standard dice), and as there are n such pairs forming these two walls it follows that $S + T = 7n$.

If n is odd then so is $7n$, which in turn means that $S + T$ is also odd. We want a tower where both S and T are odd, and so $S + T$ would in fact be even. Therefore n cannot be odd.

If n is even then so is $7n$, which in turn means that $S + T$ is also even. Therefore it may be possible to build such a tower if n is even. In fact, we will show that it is possible to build such a tower for any even value of n.

Consider stacking $n = 2k$ dice one on top of the other so that they all have the same orientation. That is to say that each vertical wall consists of n identical faces all showing the same number of dots. Next rotate the top die in the tower 180° (as you look down from above the tower). An example, with $n = 2 \times 2 = 4$, is shown in the diagram on the right.



Let R be the total number of dots visible on one of the vertical walls of this tower and let the number of dots on each of the first $2k - 1$ faces be a. The face at the top of the wall will have $7 - a$ dots. This means the total number of dots visible for this particular wall will be $R = (2k - 1)a + (7 - a)$.

If a is odd then R will be the sum of an odd number and an even number, and so will be odd. If a is even then R will be the sum of even number and an odd number, and so R will be odd. This means that no matter which wall we consider, the total number of dots visible will be odd.

We have shown that such a tower can be built whenever n is even.

B5. Charlie chooses one cell from a blank $n \times n$ square grid and shades it. The resulting grid has no lines of symmetry.

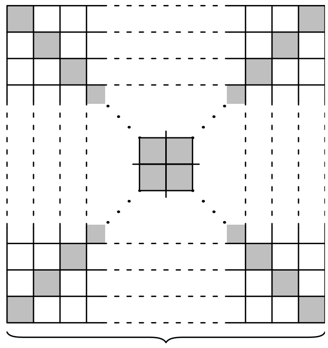
In terms of n, how many different cells could be shaded?

SOLUTION

If the cell shaded is reflected in a line of symmetry of the blank grid, it must be reflected to a different cell or else that line is a line of symmetry of the resulting grid. In addition, shading a single cell cannot introduce a new line of symmetry because reflecting in any other line will cause at least one cell to reflect to a position outside of the grid. Therefore it must be that the cell shaded is such that no line of symmetry of the blank grid passes through it.

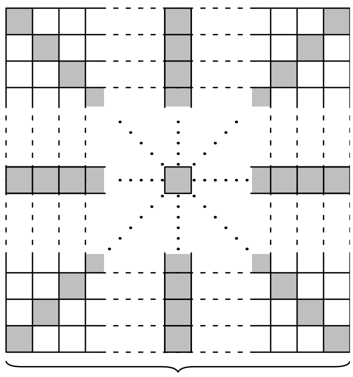
We will first count the number of cells which **do** have a line of symmetry passing through them. This count will differ depending on the parity of n, that is whether n is even or odd. We can then subtract this from the total number of cells, which is n^2 in both cases.

For n even, the only lines of symmetry of the blank $n \times n$ grid that pass through cells of the grid are the two main diagonals as pictured. Each of these lines pass through n cells, and no cell lies on both lines. Hence there are $2n$ such cells when n is even.



n even

For n odd, there are four lines of symmetry of the blank $n \times n$ grid. Two are the main diagonals and the other two are horizontal and vertical lines which pass through the centre of the grid as pictured. These four lines each pass through n cells but the centre cell lies on all of them. Hence there are $n + n + n + n - 3 = 4n - 3$ such cells when n is odd.



n odd

If n is even, then the number of cells which can be shaded in order to avoid a line of symmetry is $n^2 - 2n$.

If n is odd, then the number of cells which can be shaded in order to avoid a line of symmetry is $n^2 - (4n - 3) = n^2 - 4n + 3$.

B6. The descriptors **'even'**, **'factors of 240'**, **'multiple of 3'**, **'odd'**, **'prime'** and **'square'** are to be placed in some order as row and column headings around the grid in positions a, b, c, d, e and f.

	a	b	c
d			
e			
f			

The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are to be placed in the empty cells inside the grid so that each digit satisfies both the relevant row and column headings.

(i) Show that it is possible to complete the grid.

(ii) In how many different ways can the grid be completed?

SOLUTION

(i) Below is one possible way to complete the grid:

	even	**multiple of 3**	**odd**
factor of 240	8	6	5
prime	2	3	7
square	4	9	1

(ii) The number 8 only has two of the given properties, and so if it is to be placed into the grid then the descriptors **even** and **factors of 240** must be paired together as its column and row headings.

Whole numbers are either even or odd but not both and so the descriptors **even** and **odd** must both be column headings or both be row headings.

No whole number can be both **prime** and **square** and so these two descriptors must both be column headings or both be row headings.

This means that the three descriptors placed as column headings must be one of the two sets {**even**, **odd**, **multiple of 3**} or {**factors of 240**, **prime**, **square**} and the row headings the other.

For each set there are three choices for the descriptor which comes first, then for each of these choices there are two choices for which descriptor comes second and finally a single choice for which comes last. Therefore each of these two sets of descriptors can be arranged in $3 \times 2 \times 1 = 6$ different ways before being placed into the grid. Hence there are $2 \times 6 \times 6 = 72$ different ways to place the descriptors into the grid.

Next we need to count in how many ways the numbers can be placed into the grid.

Firstly we consider the column (or row) labelled **even**. It must contain a factor of 240, a prime and a square. Respectively these must be 8 as an even factor of 240 (it only has these two properties), 2 as the only even prime and 4 as the only even square less than 10.

Secondly we consider the column (or row) labelled **multiple of three**. It must contain a factor of 240, a prime and a square. Respectively these must be 6 as a multiple of 3 and a factor of 240, 3 as the only multiple of 3 which is prime and 9 as the only multiple of 3 less than 10 which is square.

Finally we consider the column (or row) labelled **odd**. It must contain a factor of 240, a prime and a square. Respectively these must be 5 as an odd factor of 240, 7 as the only remaining odd prime and 1 as the only remaining odd square.

Therefore once the descriptors have been placed into the grid, there is only one way to place the numbers into the grid.

In conclusion, there are 72 ways to complete the grid.